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# Information entropy of Gegenbauer polynomials

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**Abstract.** The information entropy of Gegenbauer polynomials is relevant since this is related to the angular part of the information entropies of certain quantum mechanical systems such as the harmonic oscillator and the hydrogen atom in $D$ dimensions. We give an effective method to compute the entropy for Gegenbauer polynomials with an integer parameter and obtain the first few terms in the asymptotic expansion as the degree of the polynomial tends to infinity.

## 1. Introduction and statement of results

Over the last few years much attention has been paid to the study of the information entropy

$$S_\rho = -\int \rho(x) \ln \rho(x)\, dx.$$

Probably, the most important case is when $\rho(x) = |\Psi(x)|^2$, where $\Psi(x)$ is the wavefunction of a quantum mechanical system (in the position or momentum space). For many standard models, such as the harmonic oscillator or the hydrogen atom, the wavefunction can be expressed in terms of some classical orthogonal polynomials (Gegenbauer, Laguerre, Hermite, etc) [5, 9]. This gives rise to the study of the entropy of these families, that is, of functionals of the form

$$S_n(w) = \int_\Delta q_n^2(x) \ln q_n^2(x) w(x)\, dx \tag{1}$$

where $\Delta$ is an interval of the real axis, $w$ is a weight supported on $\Delta$ and $q_n$ is the corresponding orthonormal polynomial of degree $n$.

The asymptotic properties of the sequence (1) as $n \to \infty$ have been thoroughly studied in [2, 3] for a wide class of weights $w$. Nevertheless, in practice it is very important to compute $S_n(w)$ for *each* $n \in \mathbb{N}$. In this paper we obtain explicit formulae for the entropy of Gegenbauer polynomials, i.e. polynomials

$$G_n^l(x) = g_{nl}\, x^n + \text{lower degree terms} \qquad g_{nl} > 0$$

orthonormal on $\Delta = [-1, 1]$ with respect to the probability density

$$w_l(x) = c_l (1 - x^2)^{l-1/2} \qquad x \in \Delta.$$

It is known that for $l \geqslant 1$,

$$c_l = \frac{\Gamma(l+1)}{\sqrt{\pi}\,\Gamma(l+1/2)} \qquad g_{nl} = \frac{2^n \Gamma(n+l)}{\Gamma(l+1)} \left( \frac{l(n+l)\,\Gamma(2l)}{\Gamma(n+2l)\,n!} \right)^{1/2}. \tag{2}$$

For the sake of brevity, we denote by $S_n^l$ the entropy of the polynomials $G_n^l$,

$$S_n^l = \int_\Delta (G_n^l(x))^2 \ln(G_n^l(x))^2\, w_l(x)\,\mathrm{d}x. \tag{3}$$

Expressions for $S_n^l$ have been obtained recently for $l = 0, 1, 2$. We will refer to these special cases below. Our aim is a generalization of the method proposed by one of the authors [4], in order to develop an effective procedure for computing $S_n^l$ for any $l \in \mathbb{N}$, $l \geqslant 2$.

In what follows we denote by $\mathbb{P}_n$ the space of all polynomials of degree $\leqslant n$, and $\mathbb{P} = \cup_{n \geqslant 0} \mathbb{P}_n$. In order to make the statement of the main result self-contained, we must introduce an additional piece of notation. Keeping the value $l \in \mathbb{N}$ ($l \geqslant 2$) fixed, we can generate the polynomials $P_{-1} = 0$, $P_0 = 1, \ldots, P_{2l-2}$, using the following recurrence relation:

$$P_{j+1}(x) = (2l - 2j - 3)xP_j(x) - (n+j+1)(n+2l-j-1)(1-x^2)P_{j-1}(x). \tag{4}$$

The parameters of $P_{2l-2}$ will play a special role, so we will write

$$P(x) = P_{2l-2}(x) = \alpha_{nl} \prod_{j=1}^{2l-2} (x - \xi_j). \tag{5}$$

In particular, $\alpha_{nl} \neq 0$. Additionally, define

$$H(x) = \sum_{j=0}^{2l-2} (-1)^j P_{j-1}(x) P_{2l-j-3}(x) = \beta_{nl}\, x^{2l-4} + \text{lower degree terms}. \tag{6}$$

The main result of this paper is the following:

**Theorem 1.** *For $l, n \in \mathbb{N}$, $l \geqslant 2$, let $P$ and $H$ be as defined above. Then the following formula holds:*

$$S_n^l = s_{nl} + r_{nl} \sum_{j=1}^{2l-2} (1 - \xi_j^2) \left[ \frac{H}{P'} \frac{G_{n-1}^{l+1}}{G_n^l} \right](\xi_j) \tag{7}$$

*where*

$$s_{nl} = 2\ln(g_{nl}/2^n) - \frac{n}{n+l} + 2n(n+l)\frac{\beta_{nl}}{\alpha_{nl}} + 2n \sum_{j=l}^{2l-1} \frac{1}{n+j} \tag{8}$$

$$r_{nl} = 2(n+l)\sqrt{\frac{2(l+1)n(n+2l)}{2l+1}}. \tag{9}$$

*The leading coefficients $g_{nl}$ of $G_n^l$ are given in (2).*

Interpolated values for $S_n^l$, $n = 1, \ldots, 75$, are plotted in figure 1 for $l = 2, 3, 4$.

Formulae (7)–(9) allow also to obtain a refinement of the asymptotic results of [2]. From [2, theorem 2] it follows that when $n \to \infty$, the sequence of entropies $S_n^l$ tends to the value

$$S_\infty^l = 1 + \ln \frac{\Gamma(2l)}{\Gamma(l)\Gamma(l+1)}. \tag{10}$$

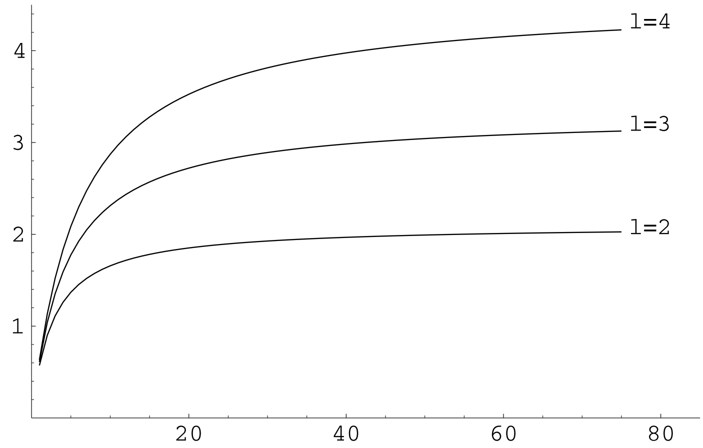


**Figure 1.** Entropy $S_n^l$, $n = 1, \ldots, 75$, for $l = 2, 3, 4$.

In order to state the result we need once again some auxiliary polynomials. For fixed $l \in \mathbb{N}$, $l \geqslant 2$, we generate $S_{-1} = 0$, $S_0 = 1, \ldots, S_{2l-2}$, but now by the following recurrence:

$$S_{j+1}(x) = (2l - 2j - 3)S_j(x) - xS_{j-1}(x) \qquad S = S_{2l-2} \tag{11}$$

and denote by $R$ the polynomial obtained as in (6), but with the polynomials $P_j$ replaced by $S_j$:

$$R(x) = \sum_{j=0}^{2l-2} (-1)^j S_{j-1}(x) S_{2l-j-3}(x). \tag{12}$$

**Theorem 2.** *For $l \geqslant 2$ the sequence of Gegenbauer entropies has the following asymptotic expansion as $n \to \infty$:*

$$S_n^l = S_\infty^l + \frac{\gamma_l}{n} + \mathrm{O}(n^{-2}) \tag{13}$$

*where*

$$\gamma_l = -2l^2 + l - 2 \sum_{j=1}^{l-1} \sqrt{\zeta_j} \frac{R}{S'}(\zeta_j) \frac{J_{l+1/2}}{J_{l-1/2}}(\sqrt{\zeta_j}). \tag{14}$$

*Here $\zeta_j$, $j = 1, \ldots, l-1$ are the zeros of $S$, and $J_\lambda$ is the Bessel function of order $\lambda$.*

Interpolated values for $n(S_n^l - S_\infty^l)$, $n = 1, \ldots, 75$, are plotted in figure 2 for $l = 2, 3, 4$. Observe that the Bessel functions in (14) can be evaluated in a finite number of terms involving trigonometric functions, since

$$J_{n+1/2}(z) = R_{n,1/2}(z) J_{1/2}(z) - R_{n-1,3/2}(z) J_{-1/2}(z)$$

where $R_{n,\nu}$ are Lommel polynomials and

$$J_{1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \sin z \qquad J_{-1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cos z$$

(see, e.g., Watson [8, sections 3.4 and 9.6]).

The structure of the paper is as follows. In section 2 we give some background material, mention some well known properties of Gegenbauer polynomials and settle the notation. In section 3 we obtain an explicit expression of the logarithmic potential for Gegenbauer polynomials, which we use in section 4 to obtain the explicit expression for the entropy given in theorem 1. The asymptotic expansion given in theorem 2 is obtained in section 5.

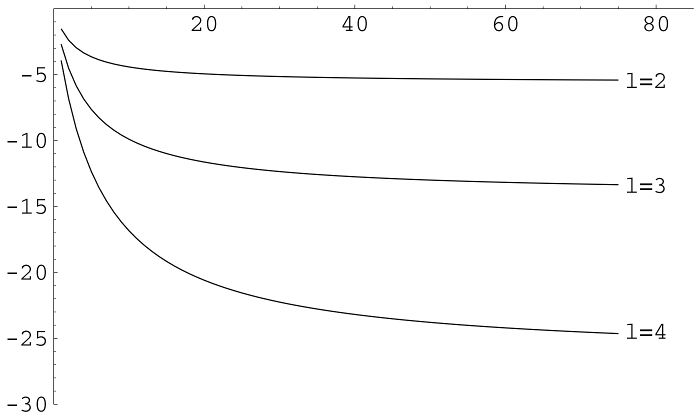


**Figure 2.** Plot of $n(S_n^l - S_\infty^l)$, $n = 1, \dots, 75$, for $l = 2, 3, 4$.

## 2. Background

We will make use of some well known facts, listed below without proof. In this section we also settle some additional notation.

The key fact on which the following computation is based is the connection between the entropy of the orthogonal polynomials and potentials, as established in [5]. If we denote by

$$V_n^l(x) = \int_\Delta \ln \frac{1}{|t-x|} |G_n^l(t)|^2 w_l(t)\, \mathrm{d}t$$

the logarithmic potential of the weight $|G_n^l(x)|^2 w_l(x)$, then by definition of $G_n^l$ we have

$$S_n^l = 2 \ln g_{nl} - 2 \sum_{j=1}^{n} V_n^l(x_j). \tag{15}$$

Here $x_j = x_{n,j}^{(l)}$ are the zeros of the polynomial $G_n^l$. In what follows we omit the indices $n$ and $l$ from the notation of the zeros of $G_n^l$, whenever this cannot lead to confusion.

If $T_n(x) = \cos(n \arccos x) = 2^{n-1}x^n + \cdots$ is the Chebyshev polynomial of the first kind, then [7, equation (4.7.14) on p 81]

$$G_n^l(x) = k_n^l T_{n+l}^{(l)}(x) \qquad k_n^l = \left[ \frac{2}{\pi} \frac{\Gamma(n+1)}{c_l(n+l)\Gamma(n+2l)} \right]^{1/2} > 0 \tag{16}$$

where $T^{(l)}$ denotes the $l$th derivative of $T$.

A consequence of a well known trigonometric identity is the following formula, which we shall call Euler's formula:

$$[T_n(x)]^2 + (1-x^2) \left[ \frac{T_n'(x)}{n} \right]^2 = 1. \tag{17}$$

We make use of the function of the second kind,

$$Q_n^l(x) = \int_\Delta \frac{T_{n+l}^{(l)}(t)}{t-x} (1-t^2)^{l-1/2}\, \mathrm{d}t$$

for $x \in \Delta$ the integral is taken in the sense of the principal value (i.e. it is the sum of the boundary values of $Q_n^l$ on $\Delta$). It is a simple exercise to prove that [1, equations (22.13.3) and (22.13.4) on p 785]

$$Q_{n+l-1}^1(x) = -\pi(n+l)T_{n+l}(x) \qquad Q_{n+l}^0(x) = \frac{\pi}{n+l}\,T'_{n+l}(x). \tag{18}$$

Finally, the recurrence relations for $Q_n^l$

$$Q_n^l(x) = (2l-3)xQ_{n+1}^{l-1}(x) - (n+2)(n+2l-2)(1-x^2)Q_{n+2}^{l-2}(x) \tag{19}$$

are a straightforward consequence of the differential equation [7, equation (4.7.5) on p 80]

$$(1-x^2)T_{n+l}^{(l)}(x) = (2l-3)xT_{n+l}^{(l-1)}(x) - (n+2)(n+2l-2)T_{n+l}^{(l-2)}(x). \tag{20}$$

Note, in particular, that $Q_n^l$ are polynomials for $n \in \mathbb{N}$ and $l$ an integer. This is, in general, not so for $l$ not an integer.

## 3. Explicit formulae for potentials

Let us introduce two auxiliary functions,

$$t_n^l(x) = \sqrt{1-x^2}\,\frac{T_{n+l}^{(l)}}{T_{n+l}^{(l-1)}}(x) \qquad q_n^l(x) = \frac{1}{\sqrt{1-x^2}}\,\frac{Q_n^l}{Q_{n+1}^{l-1}}(x)$$

and a new variable,

$$y = \frac{x}{\sqrt{1-x^2}}. \tag{21}$$

Then (19) and (20) can be rewritten as

$$t_n^l(x) = (2l-3)y - \frac{(n+2)(n+2l-2)}{t_{n+1}^{l-1}(x)} \tag{22}$$

$$q_n^l(x) = (2l-3)y - \frac{(n+2)(n+2l-2)}{q_{n+1}^{l-1}(x)}. \tag{23}$$

Denote by $p_k$ and $r_k$, $k = 1, \ldots, 2l-2$ the linearly independent solutions of the recurrence relation

$$\tau_{k+1}(y) = (2l-2k-3)y\,\tau_k(y) - (n+k+1)(n+2l-k-1)\tau_{k-1}(y) \tag{24}$$

given by their initial conditions

$$p_{-1}(y) = 0 \quad p_0(y) = 1 \qquad \text{and} \qquad r_0(y) = 0 \quad r_1(y) = 1.$$

In particular, $\deg p_k = k$, $\deg r_k = k-1$. Observe that both sequences depend on the integer parameters $l$ and $n$, in such a way that if $p_k(y) = p_k(l, n; y)$, then $r_k(y) = p_{k-1}(l-1, n+1; y)$. Whenever it cannot lead to confusion, we omit the explicit reference to these parameters in the polynomials.

Standard arguments allow us to show that for $k = 1, \ldots, l-1$, we have

$$t_n^l(x) = \frac{p_k(y)t_{n+k}^{l-k}(x) - (n+k+1)(n+2l-k-1)p_{k-1}(y)}{r_k(y)t_{n+k}^{l-k}(x) - (n+k+1)(n+2l-k-1)r_{k-1}(y)}. \tag{25}$$

The sequence $q_n^l$ also satisfies a formula analogous to (25). To be consistent with (21), denote

$$y_j = \frac{x_j}{\sqrt{1 - x_j^2}}. \tag{26}$$

Then, taking in (25) $x = x_j$ (that is, evaluating at a zero of $t_n^l$), we obtain

$$t_{n+k}^{l-k}(x_j) = (n + k + 1)(n + 2l - k - 1)\frac{p_{k-1}}{p_k}(y_j). \tag{27}$$

Now we need to prove some auxiliary results.

**Proposition 1.** *For* $0 \leqslant s \leqslant l$ *the following identities hold:*

$$\frac{T_{n+l}^{(l-s)}}{T_{n+l}}(x_j) = (n + l)\left(\sqrt{1 - x_j^2}\right)^{s-l}\frac{\Gamma(n + 2l - s)}{\Gamma(n + s + 1)}\frac{p_{s-1}}{p_{l-1}}(y_j) \tag{28}$$

$$\frac{Q_{n+s}^{l-s}}{T_{n+l}}(x_j) = \pi(-1)^{l-s}(n + l)\left(\sqrt{1 - x_j^2}\right)^{l-s-1}\frac{p_{2l-s-2}}{p_{l-1}}(y_j). \tag{29}$$

**Proof.** To obtain (28) it is sufficient to multiply the identities in (27) varying $k$ from $s$ to $l - 1$. In order to establish (29) we show that

$$q_{n+l-k}^{k}(x_j) = -\frac{p_{l+k-2}}{p_{l+k-3}}(y_j) \qquad k = 1, \ldots, l. \tag{30}$$

We proceed by induction. By (18) we have

$$q_{n+l-1}^{1}(x) = -\frac{(n + l)^2}{\sqrt{1 - x^2}}\frac{T_{n+l}}{T'_{n+l}}(x) \tag{31}$$

and using (28) with $s = l - 1$ we obtain (30) for $k = 1$.

Assume that we have established (30) for certain $k \geqslant 1$. Then, by recurrence (23),

$$q_{n+l-k-1}^{k+1}(x_j) = (2k - 1)y_j + (n + l - k + 1)(n + l + k - 1)\frac{p_{l+k-3}}{p_{l+k-2}}(y_j)$$

and (30) for $k + 1$ follows directly from the recurrence (24). Thus, equation (30) is proved for $k = 1, \ldots, l$.

Now (29) is straightforward: multiply the identities in (30) for $k = 1, \ldots, l - s$ and use (18) and (31). Note that (31) is equivalent to (29) for $s = l - 1$. □

Taking in (28) $s = l - 1$, with account of Euler's formula (17) we obtain

$$T_{n+l}^2(x_j) = \frac{p_{l-1}^2}{(n + l)^2 p_{l-2}^2 + p_{l-1}^2}(y_j). \tag{32}$$

**Proposition 2.** *For* $j = 0, \ldots, l - 1$,

$$(n + j + 1)(n + 2l - j - 1)p_{j-1}p_{2l-j-3} + p_j p_{2l-j-2} = (-1)^j p_{2l-2}. \tag{33}$$

**Proof.** Again, we proceed by induction. For $j = 0$ the identity is trivial, and for $j = 1$ it follows immediately from the recurrence (24) with $k = 2l - 3$.

Assume the assertion established up to certain $j$. If we substitute in the left-hand side of (33) the expression for $p_j$ from (24), we obtain

$$
\begin{aligned}
&(n+j+1)(n+2l-j-1)p_{j-1}p_{2l-j-3} + p_j p_{2l-j-2} \\
&\qquad = p_{j-1}((2l-2j-1)yp_{2l-j-2} + (n+j+1)(n+2l-j-1)p_{2l-j-3}) \\
&\qquad\quad -(n+j)(n+2l-j)p_{j-2}p_{2l-j-2} \\
&\qquad = -p_{j-1}p_{2l-j-1} - (n+j)(n+2l-j)p_{j-2}p_{2l-j-2}.
\end{aligned}
$$

Here we have used again the recurrence (24) with $k = 2l - j - 2$. Now the induction step is completed. □

Taking in (33) $j = l - 1$ we obtain that

$$(n+l)^2 p_{l-2}^2 + p_{l-1}^2 = (-1)^{l-1} p_{2l-2}$$

which shows that $p_{2l-2}(y)$ is non-vanishing on $\mathbb{R}$.

Now we are ready to work out a formula for the potential $V_n^l$, evaluated at the zeros of the polynomial $G_n^l$. The proof relies on the recurrence established in [4], which in our notation can be stated as follows: for $x \in \Delta$,

$$V_n^l(x) = V_{n+1}^{l-1}(x) - \frac{1}{n+2l-1} + c_l (k_n^l)^2 (1-x^2) T_{n+l}^{(l)}(x) Q_{n+1}^{l-1}(x). \tag{34}$$

The expression for the initial value $V_{n+l}^0$ was obtained in [5]:

$$V_{n+l}^0(x) = \ln 2 - \frac{1}{2(n+l)} + \frac{T_{n+l}^2(x)}{n+l} \qquad x \in \Delta. \tag{35}$$

By (34) and (35), $V_n^l \in \mathbb{P}$ for $x \in \Delta$. This property, which is only true for integer $l$, plays a key role in the proof of (7).

**Proposition 3.** *If* $G_n^l(x_j) = 0$ *then*

$$V_n^l(x_j) = \ln 2 + \frac{1}{2(n+l)} - v_{nl} + (n+l)\frac{h_{2l-4}}{p_{2l-2}}(y_j) \tag{36}$$

*where*

$$v_{nl} = \sum_{j=l}^{2l-1} \frac{1}{n+j}$$

*and*

$$h_{2l-4}(y) = \sum_{j=0}^{2l-2} (-1)^j p_{j-1}(y) p_{2l-j-3}(y) \qquad \deg h_{2l-4} = 2l-4. \tag{37}$$

**Proof.** Applying the recurrence (34) backwards in $l$ we obtain

$$V_n^l(x) = V_{n+l}^0(x) - v_{nl} + (1-x^2) \sum_{j=0}^{l-1} c_{l-j} (k_{n+j}^{l-j})^2 T_{n+l}^{(l-j)} Q_{n+j+1}^{l-j-1}(x). \tag{38}$$

Taking in (38) $x = x_k$, by means of propositions 1 and 2 and using (32) we obtain

$$V_n^l(x_k) = V_{n+l}^0(x_k) - v_{nl} + 2(n+l) \sum_{j=0}^{l-1} (-1)^j \frac{p_{j-1}p_{2l-j-3}}{p_{2l-2}}(y_k). \tag{39}$$

Moreover, by (32) and (33) identity (35) can be rewritten as

$$V^0_{n+l}(x_j) = \ln 2 + \frac{1}{2(n+l)} + (n+l)(-1)^l \frac{p^2_{l-2}}{p_{2l-2}}(y_j).$$

Clearly,

$$(-1)^l \frac{p^2_{l-2}}{p_{2l-2}} + 2\sum_{j=0}^{l-1}(-1)^j \frac{p_{j-1}p_{2l-j-3}}{p_{2l-2}} = \sum_{j=0}^{2l-2}(-1)^j \frac{p_{j-1}p_{2l-j-3}}{p_{2l-2}}$$

which establishes the assertion. □

Finally, we are going back to our variable $x$ from (21). Define

$$P_j(x) = \left(\sqrt{1-x^2}\right)^j p_j(y) \tag{40}$$

it can readily be seen that $P_j$ is a polynomial of degree $j$. Due to (24), $\{P_j\}$ satisfies the recurrence (4) with the initial condition

$$P_{-1}(x) = 0 \qquad P_0(x) = 1.$$

Now, from proposition 3 we have

$$V^l_n(x_j) = \ln 2 + \frac{1}{2(n+l)} - v_{nl} + (n+l)(1-x_j^2)\frac{H}{P}(x_j) \tag{41}$$

where $H$ is given by (6). We also follow our previous agreement to denote $P_{2l-2}$ by $P$ (see (5)).

## 4. Explicit expression for the entropy. Particular cases

Let us introduce the rational function

$$F^l_n(z) = (1-z^2)\frac{H}{P}\frac{T^{(l+1)}_{n+l}}{T^{(l)}_{n+l}}(z).$$

Then by (41)

$$\sum_{j=1}^{n} V^l_n(x_j) = n\ln 2 + \frac{n}{2(n+l)} - n\,v_{nl} + (n+l)\sum_{j=1}^{n}\operatorname*{res}_{x_j} F^l_n. \tag{42}$$

Let $\xi_j = \xi^{(l)}_{j,n}$, $j = 1, \ldots, 2l-2$, be the zeros of $P(x)$. Using the Cauchy residue theorem, we obtain

$$\sum_{j=1}^{n} V^l_n(x_j) = n\ln 2 + \frac{n}{2(n+l)} - nv_{nl} - (n+l)\operatorname*{res}_{\infty} F^l_n - (n+l)\sum_{j=1}^{2l-2}\operatorname*{res}_{\xi_j} F^l_n.$$

The residue of $F^l_n$ at infinity is $n\,\beta_{nl}/\alpha_{nl}$, where as above, $\beta_{nl}$ and $\alpha_{nl}$ are the leading coefficients of $H$ and $P$, respectively. Thus

$$\sum_{j=1}^{n} V^l_n(x_j) = n\ln 2 + \frac{n}{2(n+l)} - nv_{nl} - n(n+l)\frac{\beta_{nl}}{\alpha_{nl}} - (n+l)\sum_{j=1}^{2l-2}\operatorname*{res}_{\xi_j} F^l_n.$$

From this identity, with account of (15) and (16), the assertion of theorem 1 readily follows with $r_{nl} = 2(n+l)k^l_n/k^{l+1}_{n-1}$.

Let us consider some particular cases. Although formula (7) is valid for $l \geqslant 2$, we can apply it for $l = 1$ taking the empty sum on the right-hand side. Thus,

$$S_n^1 = s_{n1} = \frac{n}{n+1}$$

which was obtained in [9] (see also [5]).

The case $l = 2$ was studied in [4]. Now $H(x) = -1$ and

$$P(x) = (n+1)(n+3)(x^2 - \xi^2) \qquad \xi = \frac{n+2}{\sqrt{(n+1)(n+3)}}.$$

By the symmetry of $G_n^l$, equation (7) reads as

$$S_n^2 = s_{n2} - \frac{r_{n2}(1-\xi^2)G_{n-1}^3(\xi)}{(n+1)(n+3)\xi\, G_n^2(\xi)}.$$

Since

$$r_{n2} = 2(n+2)\sqrt{\frac{6n(n+4)}{5}} \qquad \text{and} \qquad g_{n2} = 2^n\sqrt{\frac{3(n+1)}{n+3}}$$

we obtain

$$S_n^2 = \ln\left(\frac{3(n+1)}{n+3}\right) + \frac{n(n^2+2n-1)}{(n+1)(n+2)(n+3)} + \frac{2}{\sqrt{(n+1)^3(n+3)^3}}\frac{T'''_{n+2}(\xi)}{T''_{n+2}(\xi)}.$$

This formula was obtained in [4] (note two misprints in that paper). We can make use of the differential equation for the Chebyshev polynomials in order to obtain the following explicit expression:

$$S_n^2 = \ln\left(\frac{3(n+1)}{n+3}\right) + \frac{n^3-5n^2-29n-27}{(n+1)(n+2)(n+3)} + \frac{1}{n+2}\left(\frac{n+3}{n+1}\right)^{n+2}.$$

## 5. Asymptotic behaviour of the entropy

In this section we are interested in the asymptotics of the potentials and the entropy of Gegenbauer polynomials as $n \to \infty$. According to theorem 1, we first need to establish the behaviour of the polynomials $P$ given by (4). Recall that $P_j$ depends both on $l$ and $n$, but we omit it from the notation.

We introduce the sequence of polynomials $S_j(z) = S_j(l, z)$, depending on the integer parameter $l$ and given by the recurrence (11), with the initial conditions $S_{-1}(l, z) = 0$ and $S_0(l, z) = 1$. Once again, considering $l$ fixed, we avoid explicit reference to it in $S_j$.

**Proposition 4.** *For* $j = 0, \ldots, 2l-2$,

$$\lim_{n\to\infty} P_j\left(1 - \frac{z}{2n^2}\right) = S_j(z) \tag{43}$$

$$\lim_{n\to\infty} \frac{1}{2n^2} P_j'\left(1 - \frac{z}{2n^2}\right) = -S_j'(z) \tag{44}$$

*locally uniformly in* $\mathbb{C}$.

**Proof.** We prove the identity (43) by induction on $j$. Then (44) readily follows from (43) by taking derivatives.

For $j = -1$ and 0 (43) holds trivially. Assume that it has been established for all natural indices not greater than $j$. By (4),

$$P_{j+1}\left(1 - \frac{z}{2n^2}\right) = (2l - 2j - 3)P_j\left(1 - \frac{z}{2n^2}\right) - \frac{(n + j + 1)(n + 2l - j - 1)}{n^2} z P_{j-1}\left(1 - \frac{z}{2n^2}\right) + \mathrm{O}(n^{-2})$$

locally uniformly in $\mathbb{C}$. According to the induction hypothesis, both polynomials $P_{j-1}$ and $P_j$ in this formula have limits given by (43). Thus, the whole right-hand side converges locally uniformly in $\mathbb{C}$. It remains to use the recurrence (11) for $S_j(z)$. □

**Remark 1.** By the symmetry of $P$ with respect to the origin,

$$\lim_{n\to\infty} P_j\left(-1 - \frac{z}{2n^2}\right) = (-1)^j S_j(-z) \tag{45}$$

$$\lim_{n\to\infty} \frac{1}{2n^2} P_j'\left(-1 - \frac{z}{2n^2}\right) = (-1)^j S_j'(-z). \tag{46}$$

Now we can study the asymptotic behaviour of the zeros $\xi_{j,n}^{(l)}$, $j = 1, \ldots, 2l - 2$, of $P(z)$. Let us recall the sequence $p_k$ generated by (24) with $p_{-1} = 0$, $p_0 = 1$. If we denote

$$p_k(y) = \sum_{j=0}^{k} a_j^k(n) y^j$$

then by (24), $p_k(-y) = (-1)^k p_k(y)$, and $a_j^k(n) = 0$ for $k - j$ odd. It is easy to prove by induction that for even $k - j$ the $a_j^k(n)$ are polynomials in $n$ of degree $k - j$. Taking into account (40) we have

$$P(x) = \sum_{j=0}^{l-1} a_{2j}^{2l-2}(n)\, x^{2j} (1 - x^2)^{l-1-j}. \tag{47}$$

Since $\deg a_{2j}^{2l-2}(n) = 2l - 2 - 2j$,

$$\frac{P(x)}{a_0^{2l-2}(n)} = (1 - x^2)^{l-1} + \mathrm{O}(n^{-2}) \tag{48}$$

locally uniformly in $\mathbb{C}$. Thus, via Hurwitz' theorem [6, corollary 4.10e on p 283], $l - 1$ zeros of $P$ tend to 1, and the remaining $l - 1$ zeros tend to $-1$. Moreover, by (43) and (45), the zeros $\xi_{j,n}^{(l)}$, $j = 1, \ldots, 2l - 2$ of $P(z)$ can be numbered in such a way that the following expression holds:

$$\xi_{j,n}^{(l)} = (-1)^j - \frac{\zeta_{j,l}(n)}{2n^2} \qquad \zeta_{2j-1,l}(n) = -\zeta_{2j,l}(n)$$

and there exist finite limits

$$\lim_{n\to\infty} \zeta_{2j,l}(n) = \zeta_j \qquad j = 1, \ldots, l - 1.$$

By proposition 4,

$$S_{2l-2}(\zeta_j) = 0.$$

We can use the well known formulae of Mehler–Heine [7, theorem 8.1.1]:

$$\lim_{n\to\infty} T_{n+l}\left(1-\frac{z}{2n^2}\right) = \cos\sqrt{z} = \sqrt{\pi/2}z^{1/4}J_{-1/2}(\sqrt{z})$$

locally uniformly in $\mathbb{C}$. Here $J_\lambda$ is the Bessel function of order $\lambda$, which can be given explicitly by

$$J_\lambda(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\lambda+k+1)}\left(\frac{z}{2}\right)^{\lambda+2k}.$$

Taking derivatives and using the well known properties of the Bessel functions, we obtain that locally uniformly in $\mathbb{C}$

$$\lim_{n\to\infty} n^{-2k}T_{n+l}^{(k)}\left(1-\frac{z}{2n^2}\right) = \sqrt{\pi/2}z^{-k/2+1/4}J_{k-1/2}(\sqrt{z}).$$

Thus, we have the asymptotic expression

$$\lim_{n\to\infty} n^{-2}\frac{T_{n+l}^{(l+1)}}{T_{n+l}^{(l)}}\left(1-\frac{z}{2n^2}\right) = z^{-1/2}\frac{J_{l+1/2}}{J_{l-1/2}}(\sqrt{z}).$$

Evaluating at the zeros of $P$ of even index and using proposition 4, we have

$$\lim_{n\to\infty} n^2(1-\xi_{2j,n}^2)\frac{H}{P'}\frac{T_{n+l}^{(l+1)}}{T_{n+l}^{(l)}}(\xi_{2j,n}) = -\frac{\sqrt{\zeta_j}}{2}\frac{R}{S'}(\zeta_j)\frac{J_{l+1/2}}{J_{l-1/2}}(\sqrt{\zeta_j}) \tag{49}$$

with

$$S(z) = S_{2l-2}(z) \qquad \text{and} \qquad R(z) = \sum_{j=0}^{2l-2}(-1)^j S_{j-1}(z)S_{2l-j-3}(z).$$

It remains to study the asymptotics of $s_{nl}$ in (8). Easy computations show that for $v_{nl}$ from proposition 3 we have the asymptotics

$$v_{nl} = \sum_{j=l}^{2l-1}\frac{1}{n+j} = \frac{l}{n} - \frac{l(3l-1)}{2n^2} + \mathrm{O}(n^{-3}). \tag{50}$$

In order to compute the leading coefficients of the polynomials $P$ and $H$, observe that

$$P_j(x) = (\sqrt{1-x^2})^j p_j\left(\frac{x}{\sqrt{1-x^2}}\right) = \sum_{k=0}^{j} a_k^j(n)x^k(\sqrt{1-x^2})^{j-k}$$

so that

$$P_j(x) = \mathrm{i}^j p_j(-\mathrm{i})x^j + \cdots$$

(where i is the imaginary unit). Taking into account that $p_{2l-2}$ and $h_{2l-4}$ are even, we have

$$\frac{\beta_{nl}}{\alpha_{nl}} = -\frac{h_{2l-4}}{p_{2l-2}}(\mathrm{i}).$$

Using (37) and taking into account the degrees of $a_j^k(n)$ as polynomials in $n$, it is not difficult to check the asymptotics

$$-\frac{h_{2l-4}}{p_{2l-2}}(\mathrm{i}) = \frac{1}{a_0^{2l-2}(n)}\sum_{j=0}^{l-2}a_0^{2j}(n)a_0^{2l-2j-4}(n) + \mathrm{O}(n^{-4})$$

so that by (33),

$$-\frac{h_{2l-4}}{p_{2l-2}}(\mathrm{i}) = -\sum_{j=0}^{l-2} \frac{1}{(n+2j+2)(n+2l-2j-2)} + \mathrm{O}(n^{-4}).$$

Then,

$$n(n+l)\frac{\beta_{nl}}{\alpha_{nl}} = -l + 1 + \frac{l(l-1)}{n} + \mathrm{O}(n^{-2}). \tag{51}$$

By (2) and (16),

$$2\ln\left(g_{nl}/2^n\right) = \ln\left(\frac{\Gamma(2l)}{\Gamma(l)\Gamma(l+1)}\right) + \ln\left(\frac{\Gamma(n+l)\Gamma(n+l+1)}{\Gamma(n+1)\Gamma(n+2l)}\right)$$

and Stirling's formula yields

$$2\ln g_{nl}/2^n = \ln\frac{\Gamma(2l)}{\Gamma(l)\Gamma(l+1)} - \frac{l(l-1)}{n} + \mathrm{O}(n^{-2}). \tag{52}$$

Gathering (49)–(52), the assertion of theorem 2 follows.


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